SECTION – A

1. The least value of |z| where z is complex number which satisfies the inequality exp $\left(\frac{(|z|+3)(|z|-1)}{|z|+1}\log_e 2\right) \geq \log_{\sqrt{2}}\left|5\sqrt{7}+9i\right|, i=\sqrt{-1}$ is equal to :

(1) 2

(2) 3

(3) 8

(4) $\sqrt{5}$

Ans. **(2)**

Sol. $2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq 2^3 \Rightarrow \frac{(|z|+3)(|z|-1)}{|z|+1} \geq 3$

$\Rightarrow |z|^2 + 2|z| - 3 \geq 3|z| + 3$

$\Rightarrow |z|^2 - |z| - 6 \geq 0$

$(|z|-3)(|z|+2) \geq 0$

$|z|_{min} = 3$

2. Let $f : S \rightarrow S$ where S = (0, ∞) be a twice differentiable function such that $f(x+1) = xf(x)$. If g : S → R be defined as $g(x) = \log_e f(x)$, then the value of |g"(5) – g"(1)| is equal to :

(1) $\frac{197}{144}$

(2) $\frac{187}{144}$

(3) $\frac{205}{144}$

(4) 1

Ans. **(3)**

Sol. $f(x+1) = xf(x)$

$g(x+1) = \log_e(f(x+1))$

$g(x+1) \log_e x + \log f(x)$

$g(x+1) - g(x) = \log_e x$

$g''(x+1) - g''(x) = -\frac{1}{x^2}$

$g''(2) - g''(1) = -1$

$g''(3) - g''(2) = -\frac{1}{4}$

$g''(4) - g''(3) = -\frac{1}{9}$

$g''(5) - g''(4) = -\frac{1}{16}$

$g''(5) - g''(1) = -\left[1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\right]$

$\left|g''(5)-g''(1)\right| = \left[\frac{144+36+16+9}{16\times 9}\right] = \left[\frac{205}{16\times 9}\right]$

3. If y = y(x) is the solution of the differential equation $\frac{dy}{dx}+(\tan x)y = \sin x, 0 \le x \le \frac{\pi}{3}$, with y(0) = 0, then $y\left(\frac{\pi}{4}\right)$ equal to :

(1) $\log_e 2$

(2) $\frac{1}{2}\log_e 2$

(3) $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$

(4) $\frac{1}{4}\log_e 2$

Ans. **(3)**

Sol. I.f. = $e^{\int \tan x dx}$

= $e^{\ln[\sec x]}$

= secx

Solution of the equation

y(secx) = $\int (\sin x)(\sec x)dx$

$\Rightarrow \frac{y}{\cos x} = \ell n(\sec x)+c$

Put x = 0, c = 0

$\therefore$ y = cosx $\ell n(\sec x)$

put x = $\pi/4$

$y = \frac{1}{\sqrt{2}} \ln \sqrt{2} = \frac{1}{2\sqrt{2}} \ell n2$

$y = \frac{\ln 2}{2\sqrt{2}}$

4. If the foot of the perpendicular from point (4, 3, 8) on the line L_1 : $\frac{x-a}{\ell} = \frac{y-2}{3} = \frac{z-b}{4}$, $\ell \neq 0$ is (3, 5, 7), then the shortest distance between the line L_1 and line L_2 : $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is equal to :

(1) $\sqrt{\frac{2}{3}}$

(2) $\frac{1}{\sqrt{3}}$

(3) $\frac{1}{2}$

(4) $\frac{1}{\sqrt{6}}$

Ans. **(4)**

Sol. (3, 5, 7) lie on given line L_1

$\frac{3-a}{\ell} = \frac{3}{3} = \frac{7-b}{4}$

$\frac{7-b}{4} = 1 \Rightarrow b = 3$

$\frac{3-a}{\ell} = 1 \Rightarrow 3 - a = \ell$

A (4, 3, 8)

B (3, 5, 7)

DR'S of AB = (1, –2, 1)

AB $\perp$ line L_1

$(1)(\ell) + (-2)(3) + 4(1) = 0$

$\Rightarrow \ell = 2$

a = 1

a = 1, b = 3, ℓ = 2

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

$$\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

$$\text{S.D.} = \frac{\begin{vmatrix}1 & 2 & 2\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}}{\left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}\right\|}=\frac{1}{\sqrt{6}}$$

5. If (x, y, z) be an arbitrary point lying on a plane P which passes through the points (42, 0, 0), (0, 42, 0) and (0, 0, 42), then the value of the expression

$$3+\frac{x-11}{(y-19)^2(z-12)^2}+\frac{y-19}{(x-11)^2(z-12)^2}+\frac{z-12}{(z-11)^2(y-19)^2}-\frac{x+y+z}{14(x-11)(y-19)(z-12)}$$ is equal to :

(1) 3

(2) 0

(3) 39

(4) –45

Ans. **(1)**

Sol. equation of plane x + y + z = 42

Let pt. on plane x = 10, y = 21, z = 11

$$3+\frac{(-1)}{(4)(1)}+\frac{(2)}{(1)(1)}+\frac{(-1)}{(1)(4)}-\frac{42}{14(-1)(2)(-1)}$$

$$3-\frac{1}{4}+2-\frac{1}{4}-\frac{3}{2} = 3$$

6. Consider the integral

$$I=\int_0^{10}\frac{[x]e^{[x]}}{e^{x-1}}dx$$

Where [x] denotes the greatest integer less than or equal to x. Then the value of I is equal to :

(1) 45 (e – 1)

(2) 45 (e + 1)

(3) 9(e – 1)

(4) 9(e + 1)

Ans. **(1)**

Sol. $I = \int_0^{10} [x] \cdot e^{[x]+1-x} dx$

$= \int_1^2 e^{2-x} dx + \int_2^3 2 \cdot e^{3-x} dx + \int_3^4 3 \cdot e^{4-x} dx + \ldots\ldots + \int_9^{10} 9e^{10-x} dx$

$= -\{(1-e) + 2(1-e) + 3(1-e) + \ldots\ldots + 9(1-e)\}$

$= 45(e-1)$

7. Let A (–1, 1), B (3, 4) and C(2, 0) be given three points. A line y = mx, m > 0, intersects lines AC and BC at point P and Q respectively. Let A_1 and A_2 be the areas of ΔABC and ΔPQC respectively, such that $A_1 = 3A_2$, then the value of m is equal to :

(1) $\frac{4}{15}$

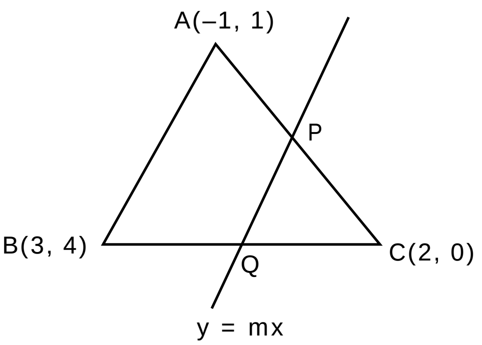
(2) 1

(3) 2

(4) 3

Ans. **(2)**

Sol.



$$A_1 = \Delta ABC = \frac{1}{2}\left\| \begin{matrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 4 & 1 \end{matrix} \right\|$$

$$A_1 = \frac{13}{2}$$

Equation of line AC is $y - 1 = -\frac{1}{3}(x + 1)$

solve it with line y = mx, we get $P\left(\frac{2}{3m+1}, \frac{2m}{3m+1}\right)$

Equation of line BC is $y - 0 = 4(x - 2)$

Solve it with line y = mx, we get $Q\left(\frac{-8}{m-4}, \frac{-8m}{m-4}\right)$

$$A_2 = \text{Area of } \Delta PQC = \frac{1}{2}\left\|\begin{matrix} 2 & 0 & 1 \\ \frac{2}{3m+1} & \frac{2m}{3m+1} & 1 \\ \frac{-8}{m-4} & \frac{-8m}{m-4} & 1 \end{matrix}\right\| = \frac{A_1}{3} = \frac{13}{6}$$

$$= \frac{1}{2}\left(2\left(\frac{2m}{3m+1} + \frac{8m}{m-4}\right) - 1\left(\frac{-16m}{(3m+1)(m-4)} + \frac{16m}{(3m+1)(m-4)}\right)\right)$$

$$= \pm\frac{13}{6}$$

$$\frac{26m^2}{3m^2 - 11m - 4} = \pm\frac{13}{6}$$

$$\Rightarrow 12m^2 = \pm\left(3m^2 - 11m - 4\right)$$

taking +ve sign

$9m^2 + 11m + 4 = 0$ (Rejcted $\because$ m is imaginary)

taking –ve sing

$15\ m^2 - 11m - 4 = 0$

$m = 1, -\frac{4}{15}$

8. Let f be a real valued function, defined on R–{–1, 1} and given by

$f(x) = 3\log_e\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1}$.

Then in which of the following intervals, function $f(x)$ is increasing ?

(1) $(-\infty, -1) \cup \left(\left[\frac{1}{2}, \infty\right) - \{1\}\right)$

(2) $\left(-1, \frac{1}{2}\right]$

(3) $(-\infty, \infty) - \{-1, 1\}$

(4) $\left(-\infty, \frac{1}{2}\right] - \{-1\}$

Ans. (1)

Sol. $f'(x) = \left(\frac{x+1}{x-1} \times \frac{x+1-(x-1)}{(x+1)^2}\right)3 + \frac{2}{(x-1)^2} = \frac{6}{(x-1)(x+1)} + \frac{2}{(x-1)^2}$

$= \frac{2}{(x-1)}\left(\frac{3}{x+1} + \frac{1}{x+1}\right) = \frac{4(2x-1)}{(x+1)(x-1)^2}$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -1 & & \frac{1}{2} & \end{array}$$

$x \in (-\infty, -1) \cup \left[\frac{1}{2}, \infty\right) - \{1\}$

9. Let the lengths of intercepts on x-axis and y-axis made by the circle $x^2 + y^2 + ax + 2ay + c = 0$, $(a<0)$ be $2\sqrt{2}$ and $2\sqrt{5}$, respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line $x + 2y = 0$, is equal to :

(1) $\sqrt{10}$

(2) $\sqrt{6}$

(3) $\sqrt{11}$

(4) $\sqrt{7}$

Ans. **(2)**

Sol. $2\sqrt{a^2/4 - c} = 2\sqrt{2}$

$\sqrt{a^2 - 4c} = 2\sqrt{2}$

$a^2 - 4c = 8$ ... (1)

$2\sqrt{a^2 - c} = 2\sqrt{5}$

$a^2 - c = 5$ ...(2)

(2) – (1)

$3c = -3a \Rightarrow c = -1$

$a^2 = 4 \Rightarrow a = -2$

$x^2 + y^2 - 2x - 4y - 1 = 0$

Equation of tangent $2x - y + \lambda = 0$

$\therefore p = r$

$$\left|\frac{2-2+\lambda}{\sqrt{5}}\right| = \sqrt{6}$$

$\Rightarrow \lambda = \pm\sqrt{30}$

$\therefore$ tangent $2x - y \pm \sqrt{30} = 0$

Distance from origin = $\frac{\sqrt{30}}{\sqrt{5}} = \sqrt{6}$

10. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

(1) $\frac{4}{9}$

(2) $\frac{9}{56}$

(3) $\frac{3}{7}$

(4) $\frac{11}{27}$

Ans. (1)

Sol. Total case = $6\underline{|6}$

Fav. case = (0, 1, 2, 3, 4, 5) + (0, 1, 2, 4, 5, 6) + (1, 2, 3, 4, 5, 6)

= $5\underline{|5} + 5\underline{|5} + \underline{|6}$

= 1920

Probability = $\frac{1920}{6\underline{|6}} = \frac{4}{9}$

11. Let $\alpha \in R$ be such that the function $f(x)=\begin{cases}\dfrac{\cos^{-1}\left(1-\{x\}^2\right)\sin^{-1}\left(1-\{x\}\right)}{\{x\}-\{x\}^3} & x\neq 0\\ \alpha & x=0\end{cases}$ is

Continuous at x = 0, where {x} = x–[x],[x] is the greatest integer less than or equal to x. Then :

(1) $\alpha=\dfrac{\pi}{4}$

(2) No such α exists

(3) α = 0

(4) $\alpha=\dfrac{\pi}{\sqrt{2}}$

Ans. (2)

Sol. RHL = $\lim\limits_{x\to 0^+}\dfrac{\cos^{-1}\left(1-x^2\right)\sin^{-1}\left(1-x\right)}{x\left(1-x^2\right)}=\dfrac{\pi}{2}\lim\limits_{x\to 0^+}\dfrac{\cos^{-1}\left(1-x^2\right)}{x}$

$=\dfrac{\pi}{2}\lim\limits_{x\to 0^+}\dfrac{-1}{\sqrt{1-\left(1-x^2\right)^2}}\left(-2x\right)$ (L' Hospital Rule)

$=\pi\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{2x^2-x^4}}=\pi\lim\limits_{x\to 0^+}\dfrac{1}{\sqrt{2-x^2}}=\dfrac{\pi}{\sqrt{2}}$

LHL = $\lim\limits_{x\to 0^-}\dfrac{\cos^{-1}\left(1-(1+x)^2\right)\sin^{-1}\left(-x\right)}{(1+x)-(1+x)^3}=\dfrac{\pi}{2}\lim\limits_{x\to 0^-}\dfrac{\sin^{-1}x}{(1+x)\left[(1+x)^2-1\right]}=\dfrac{\pi}{2}\lim\limits_{x\to 0^-}\dfrac{\sin^{-1}x}{x^2+2x}$

$=\dfrac{\pi}{2}\left(\dfrac{1}{2}\right)=\dfrac{\pi}{4}$

As LHL $\neq$ RHL so f(x) is not continuous at x = 0

12. The maximum value of $f(x)=\begin{vmatrix}\sin^2 x & 1+\cos^2 x & \cos 2x\\ 1+\sin^2 x & \cos^2 x & \cos 2x\\ \sin^2 x & \cos^2 x & \sin 2x\end{vmatrix}$, $x\in R$ is :

(1) $\sqrt{7}$

(2) $\sqrt{5}$

(3) 5

(4) $\dfrac{3}{4}$

Ans. **(2)**

Sol. $C_1 \to C_1 + C_2$

$$\begin{vmatrix} 2 & 1+\cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$= (-1)[2\sin 2x - \cos 2x] = \cos 2x - 2\sin 2x$

maximum value = $\sqrt{5}$

13. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the number of triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta-\alpha)$ is equal to:

(1) 1890

(2) 795

(3) 717

(4) 1173

Ans. **(3)**

Sol. $\alpha = {}^6C_1\,{}^7C_1\,{}^9C_1 + {}^5C_1\,{}^7C_1\,{}^9C_1 + {}^5C_1\,{}^6C_1\,{}^9C_1 + {}^5C_1\,{}^6C_1\,{}^7C_1 = 378 + 315 + 270 + 210 = 1173$

$\beta = {}^5C_1\,{}^6C_1\,{}^7C_1\,{}^7C_1\,{}^9C_1 = 1890$

$\Rightarrow \beta - \alpha = 1890 - 1173 = 717$

14. Let C be the locus of the mirror image of a point on the parabola $y^2 = 4x$ with respect to the line $y = x$. Then the equation of tangent to C at P(2, 1) is :

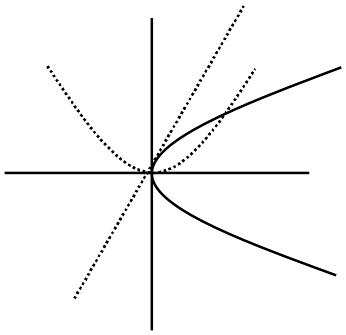
(1) $2x + y = 5$

(2) $x + 2y = 4$

(3) $x + 3y = 5$

(4) $x - y = 1$

Ans. **(4)**



Sol. Image of $y^2 = 4x$ w.r.t. $y = x$ is $x^2 = 4y$

tangent from (2, 1)

$xx_1 = 2(y + y_1)$

$2x = 2(y + 1)$

$x = y + 1$

15. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy $\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x$ is equal to :

(1) 1

(2) 2

(3) 3

(4) 0

Ans. **(3)**

Sol. Taking sine both sides

$$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}} = x$$

$$\Rightarrow 3x\sqrt{25-16x^2} = 25x - 4x\sqrt{25-9x^2}$$

$$\Rightarrow x = 0 \text{ or } 3\sqrt{25-16x^2} = 25 - 4\sqrt{25-9x^2}$$

$$\Rightarrow 9\left(25-16x^2\right) = 625 - 200\sqrt{25-9x^2} + 16\left(25-9x^2\right)$$

$$\Rightarrow 200\sqrt{25-9x^2} = 800$$

$$\Rightarrow \sqrt{25-9x^2} = 4$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x \pm 1$$

$\therefore$ Total number of solution = 3

16. Let C_1 be the curve obtained by solution of differential equation $2xy\frac{dy}{dx} = y^2 - x^2, x > 0$ Let the curve C_2 be the solution of $\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}$, If both the curves pass through (1, 1) then the area enclosed by the curves C_1 and C_2 is equal to :

(1) $\frac{\pi}{2} - 1$

(2) $\frac{\pi}{4} + 1$

(3) $\pi - 1$

(4) $\pi + 1$

Ans. (1)

Sol.

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put y = vx

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{(v^2 + 1)}{2v}$$

$$\Rightarrow \frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$$

$$\ell n(v^2 + 1) = -\ell nx + \ell nc \Rightarrow v^2 + 1 = \frac{c}{x}$$

$$\Rightarrow \frac{y^2}{x^2} + 1 = \frac{c}{x} \Rightarrow x^2 + y^2 = cx$$

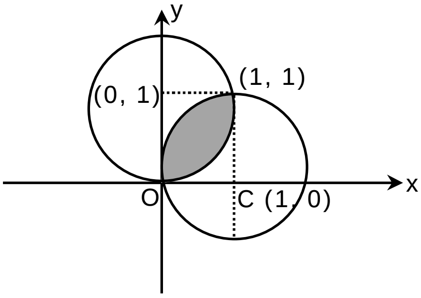
If pass through (1,1)

$\therefore\ x^2 + y^2 - 2x = 0$

Similarly for second differential equation $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Equation of curve is $x^2 + y^2 - 2y = 0$

Now required area is



$$= \left(\frac{1}{4}\times\pi\times1^2-\frac{1}{2}\times1\times1\right)\times2$$

$$= \left(\frac{\pi}{2}-1\right) \text{ sq. units}$$

17. Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=2\hat{i}-3\hat{j}+5\hat{k}$. If $\vec{r}\times\vec{a}=\vec{b}\times\vec{r}, \vec{r}\cdot\left(\alpha\hat{i}+2\hat{j}+\hat{k}\right)=3$ and $\vec{r}\cdot\left(2\hat{i}+5\hat{j}-\alpha\hat{k}\right)=-1, \alpha\in R$, then the value of $\alpha+\left|\vec{r}\right|^2$ is equal to :

(1) 11

(2) 15

(3) 9

(4) 13

Ans. **(2)**

Sol. $\vec{r}\times\vec{a}=-\vec{r}\times\vec{b}$

$\vec{r}\times\left(\vec{a}+\vec{b}\right)=0 \qquad \left(\vec{a}+\vec{b}=3\hat{i}-\hat{j}+2\hat{k}\right)$

$\vec{r}\ ||\left(\vec{a}+\vec{b}\right)$

$\vec{r}=\lambda\left(\vec{a}+\vec{b}\right)$

$\because\ \vec{r}\cdot\left(2\hat{i}+5\hat{j}-\alpha\hat{k}\right)=-1$

$\lambda\left[3\hat{i}-\hat{j}+2\hat{k}\right]\cdot\left[2\hat{i}+5\hat{j}-\alpha\hat{k}\right]=-1$

$\Rightarrow \lambda(6-5-2\alpha)=-1$

$\lambda(1-2\alpha)=-1 \qquad \text{...(1)}$

$\vec{r}\cdot\left(\alpha\hat{i}+2\hat{j}+\hat{k}\right)=3$

$\lambda\left(3\hat{i}-\hat{j}+2\hat{k}\right)\cdot\left(\alpha\hat{i}+2\hat{j}+\hat{k}\right)=3$

$\Rightarrow \lambda[3\alpha-2+2]=3 \Rightarrow \lambda\alpha=1 \qquad \text{...(2)}$

(1) & (2)

$$\lambda\left[1-\frac{2}{\lambda}\right]=-1$$

$\lambda - 2 = -1 \Rightarrow \lambda = 1 \quad \alpha = 1$

$\vec{r} = 3\hat{i} - \hat{j} + 2\hat{k}$

$\alpha + |\vec{r}|^2 \Rightarrow 1 + 14 = 15$

18. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)\,dx = 1$ and $P(x)$ leaves remainder 5 when it is divided by $(x - 2)$. Then the value of $9(b+c)$ is equal to :

(1) 7

(2) 11

(3) 15

(4) 9

Ans. **(1)**

Sol. $(x - 2)Q(x) + 5 = x^2 + bx + c$

Put $x = 2$

$5 = 2b+c+4$ ...(1)

$\int_0^1 \left(x^2 + bx + c\right) dx = 1$

$\Rightarrow \frac{1}{3} + \frac{b}{2} + c = 1$

$\frac{b}{2} + c = \frac{2}{3}$ ...(2)

Solve (1) & (2)

$b = \frac{2}{9}$

$c = \frac{5}{9}$

$9(b+c) = 7$

19. If the points of intersections of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = 4b$, $b > 4$ lie on the curve $y^2 = 3x^2$, then b is equal to :

(1) 5

(2) 6

(3) 12

(4) 10

Ans. (3)

Sol. $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ ...(1)

$x^2+y^2=4b$ ...(2)

$y^2=3x^2$ ...(3)

From eq (2) and (3) $x^2 = b$ and $y^2 = 3b$

From equation (1) $\frac{b}{16}+\frac{3b}{b^2}=1$

$\Rightarrow b^2+48=16b$

$\Rightarrow b=12$

20. Let A = {2, 3, 4, 5......,30} and '$\simeq$' be an equivalence relation on A×A, defined by (a, b) $\simeq$ (c, d), if and only if ad = bc. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to :

(1) 7

(2) 5

(3) 6

(4) 8

Ans. (1)

Sol. ad = bc

(a, b) R (4, 3) $\Rightarrow$ 3a = 4b

$a = \frac{4}{3}b$

b must be multiple of 3

b = {3, 6, 9 ......30}

(a, b) = (4, 3), (8, 6), (12, 9) (16, 12), (20, 15), (24, 18), (28, 21)

$\Rightarrow$ 7 ordered pair

SECTION – B

1. Let $\vec{c}$ be a vector perpendicular to the vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$. If $\vec{c}\cdot\left(\hat{i}+\hat{j}+3\hat{k}\right)-8$ then the value of $\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$ is equal to ....

Ans. **(28)**

Sol. $\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\left[\vec{c}\,\vec{a}\,\vec{b}\right]$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -1\\ 1 & 2 & 1\end{vmatrix}=\left(3,-2,1\right)$$

$\vec{c}\perp\vec{a},\vec{c}\perp\vec{b}\Rightarrow C\,||\,\vec{a}\times\vec{b}$

$\vec{c}=\lambda\left(\vec{a}\times\vec{b}\right)$

$\Rightarrow\ \vec{c}=\lambda\left(3\hat{i}-2\hat{j}+\hat{k}\right)$

$\vec{c}\left(\hat{i}+\hat{j}+3\hat{k}\right)=8$

$\Rightarrow 3\lambda-2\lambda+3\lambda=8$

$\Rightarrow 4\lambda=8\Rightarrow\lambda=2$

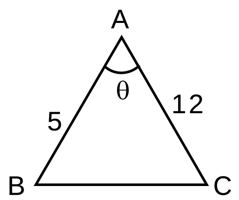
$\vec{c}=6\hat{i}-4\hat{j}+2\hat{k}$

$$\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\left[\vec{c}\,\vec{a}\,\vec{b}\right]=\begin{vmatrix}6 & -4 & 2\\ 1 & 1 & -1\\ 1 & 2 & 1\end{vmatrix}$$

$\Rightarrow 18+8+2=28$

2. In ΔABC, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of ΔABC is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of ΔABC, then the value of 2R + r (in cm) is equal to ....

Ans. **(15)**

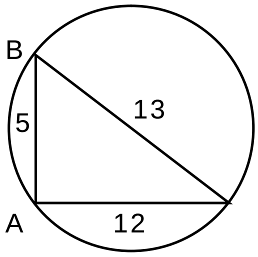


Sol.

Area = $\frac{1}{2}(5)(12)\sin\theta = 30$

$\sin\theta = 1\Rightarrow\theta=\frac{\pi}{2}$

Δ is right angle Δ



$r = (s-a)\tan\frac{A}{2}$

$r = (s-a)$

$r = (s-a) \quad (a = 2R)$

$2R + r = s$

$2R + r = \frac{30}{2} = 15$

3. Consider the statistics of observations as follows:

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is $\frac{17}{9}$, then the value of n is equal to ....

Ans. **(5)**

Sol. For group-1 : $\frac{\sum x_i}{10} = 2 \Rightarrow \sum x_i = 20$

$$\frac{\sum x_i}{10} - (2)^2 = 2 \Rightarrow \sum x_i^2 = 60$$

For group-2 : $\frac{\sum y_i}{n} = 3 \Rightarrow \sum y_i = 3n$

$$\frac{\sum y_i^2}{n} - 3^2 = 1 \Rightarrow \sum y_i^2 = 10n$$

Now, combined variance

$$\sigma^2 = \frac{\sum\left(x_i^2 + y_i^2\right)}{10+n} - \left(\frac{\sum\left(x_i + y_i\right)}{10+n}\right)^2$$

$$\Rightarrow \frac{17}{9} = \frac{60+10n}{10+n} - \frac{(20+3n)^2}{(10+n)^2}$$

$$\Rightarrow 17\left(n^2 + 20n + 100\right) = 9\left(n^2 + 40n + 200\right)$$

$$\Rightarrow 8n^2 - 20n - 100 = 0$$

$$\Rightarrow 2n^2 - 5n - 25 = 0 \Rightarrow n = 5$$

4. Let

$S_n(x) = \log_{a^{\frac{1}{2}}} x + \log_{a^{\frac{1}{3}}} x + \log_{a^{\frac{1}{6}}} x + \log_{a^{\frac{1}{11}}} x + \log_{a^{\frac{1}{18}}} x + \log_{a^{\frac{1}{27}}} x + ...$ up to n-terms,

Where a >1. If $S_{24}(x) = 1093$ and $S_{12}(2x) = 265$, the value of a is equal to ....

Ans. **(16)**

Sol. $S_n(x) = \log_a x^2 + \log_a x^3 + \log_a x^6 + \log_a x^{11}$

$S_n(x) = 2\log_a x + 3\log_a x + 6\log_a x + 11\log_a x + ...$

$S_n(x) = \log_a x(2 + 3 + 6 + 11 + ...)$

$S_r = 2 + 3 + 6 + 11$

General term $T_r = r^2 - 2r + 3$

$$S_n(x) = \sum_{r=1}^{n} \log_a x\left(r^2 - 2r + 3\right)$$

$$S_{24}(x) = \sum_{r=1}^{24} \log_a x\left(r^2 - 2r + 3\right)$$

$$S_{24}(x) = \log_a \sum_{r=1}^{24} \left(r^2 - 2r + 3\right)$$

$1093 = 4372 \log_a x$

$\log_a x = \frac{1}{4}$

$x = a^{1/4}$

$$S_{12}(2x) = \log_a(2x)\sum_{r=1}^{12}\left(r^2 - 2r + 3\right)$$

$265 = 530 \log_a(2x)$

$\log_a(2x) = \frac{1}{2}$

$2x = a^{1/2}$

After solving (i) and (ii), we get

$a^{1/4} = 2$

$a = 16$

5. Let n be a positive integer. Let $A = \sum_{k=0}^{n} (-1)^k \, {}^nC_k \left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$ If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to ....

Ans. **(6)**

Sol. $A = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n + \left(\frac{1}{32}\right)^n$

$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \frac{1}{2^{5n}}$

$$=\frac{1}{2^n}\left[\frac{1-\left(\frac{1}{2^n}\right)^5}{1-\frac{1}{2^n}}\right]$$

$$A=\frac{2^{5n}-1}{2^{5n}(2^n-1)}$$

$$63A=\frac{63\left(2^{5n}-1\right)}{2^{5n}(2^n-1)}$$

$$\frac{63}{2^n-1}\left(1-\frac{1}{2^{5n}}\right)=63A=\left(1-\frac{1}{2^{30}}\right)$$

$$=\frac{63}{2^n-1}\left(1-\frac{1}{2^{5n}}\right)=\left(1-\frac{1}{2^{30}}\right)$$

$$n=6$$

6. Let $f: R \to R$ and $g: R \to R$ be defined as

$f(x)=\begin{cases} x+a, x<0 \\ |x-1|, x \geq 0 \end{cases}$ and $g(x)\begin{cases} x+1, x<0 \\ (x-1)^2+b, x \geq 0 \end{cases}$

Where a, b are non-negative real numbers. If (gof) (x) is continuous for all $x \in R$, then a + b is equal to ....

Ans. **(1)**

Sol.

$$g[f(x)]=\begin{cases} f(x)+1 & f(x)<0 \\ (f(x)-1)^2+b & f(x)\geq 0 \end{cases}$$

$$g[f(x)]=\begin{cases} x+a+1 & x+a<0 \ \&\ x<0 \\ |x-1|+1 & |x-1|<0 \ \&\ x\geq 0 \\ (x+a-1)^2+b & x+a\geq 0 \ \&\ x<0 \\ (|x-1|-1)^2+b & |x-1|\geq 0 \ \&\ x\geq 0 \end{cases}$$

$$g[f(x)]=\begin{cases} x+a+1 & x\in(-\infty,-a) \ \&\ x\in(-\infty,0) \\ |x-1|+1 & x\in\phi \\ (x+a-1)^2+b & x\in[-a,\infty) \ \&\ x\in(-\infty,0) \\ (|x-1|-1)^2+b & x\in R \ \&\ x\in[0,\infty) \end{cases}$$

$$g[f(x)]=\begin{cases} x+a+1 & x\in(-\infty,-a) \\ (x+a-1)^2+b & x\in[-a,0) \\ (|x-1|-1)^2+b & x\in[0,\infty) \end{cases}$$

g(f(x)) is continuous

at x = –a & at x = 0

1 = b + 1 & $(a-1)^2 + b = b$

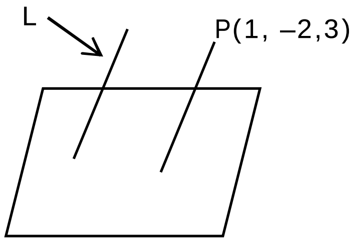
b = 0 & a = 1

$\Rightarrow$ a + b = 1

7. If the distance of the point (1, –2, 3) from the plane x + 2y – 3z + 10 = 0 measured parallel to the line, $\frac{x-1}{3}=\frac{2-y}{m}=\frac{z+3}{1}$ is $\sqrt{\frac{7}{2}}$, then the value of |m| is equal to ....

Ans. **(2)**

Sol.



$$\frac{x-1}{3}=\frac{y+2}{-m}=\frac{z-3}{1}=\lambda$$

Pt. Q $(3\lambda+1,-m\lambda-2,\lambda+3)$ lie on plane

$$(3\lambda+1)+2(-m\lambda-2)-3(\lambda+3)+10=0$$

$$\Rightarrow 3\lambda-2m\lambda-3\lambda+1-4-9+10=0$$

$$\Rightarrow -2m\lambda=2$$

$$m\lambda = -1 \Rightarrow \lambda = -\frac{1}{m}$$

$$Q\left[-\frac{3}{m}+1,-1,-\frac{1}{m}+3\right]$$

$$PQ=\sqrt{\frac{7}{2}}$$

$$\sqrt{\left(-\frac{3}{m}\right)^2+1+\left(-\frac{1}{m}\right)^2}=\sqrt{\frac{7}{2}}$$

$$\Rightarrow \frac{10+m^2}{m^2}=\frac{7}{2}$$

$$\Rightarrow 20 + 2m^2 = 7m^2$$

$m^2 = 4 \quad \Rightarrow |m| = 2$

8. Let $\frac{1}{16}$, a and b be in G.P. and $\frac{1}{a}$, $\frac{1}{b}$, 6 be in A.P. where a, b, > 0. Then 72(a+b) is equal to ....

Ans. **(14)**

Sol. $a^2 = \frac{b}{16}$ and $\frac{2}{b}=\frac{1}{a}+6$

Solving, we get $a = \frac{1}{12}$ or $a = -\frac{1}{4}$ [rejected]

if $a = \frac{1}{12} \Rightarrow b = \frac{1}{9}$

$\therefore 72(a+b) = 72\left(\frac{1}{12}+\frac{1}{9}\right) = 14$

9. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be two 2×1 matrices with real entries such that A = XB, where x = $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$, and $k\in R$. If $a_1^2 + a_2^2 = \frac{2}{3}\left(b_1^2 + b_2^2\right)$ and $\left(k^2+1\right)b_2^2 \neq -2b_1b_2$ then the value of k is ....

Ans. **(1)**

Sol.

$XB = A$

$\frac{1}{\sqrt{3}}\begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$b_1 - b_2 = \sqrt{3}a_1 \Rightarrow 3a_1^2 = b_1^2 + b_2^2 - 2b_1b_2$

$b_1 + kb_2 = \sqrt{3}a_2 \Rightarrow 3a_2^2 = b_1^2 + k^2b_2^2 + 2kb_1b_2$

$3\left(a_1^2 + a_2^2\right) = 2b_1^2 + \left(k^2+1\right)b_2^2 + 2b_1b_2\,(k-1)$

$\left(k^2-1\right)b_2^2 + 2b_1b_2\,(k-1)$

$(k-1)\left((k+1)b_2^2 + 2b_1b_2\right) = 0$

$\Rightarrow k = 1$

10. For real number α, β, γ and δ, if

$$\int \frac{\left(x^2-1\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right)}{\left(x^4+3x^2+1\right)\tan^{-1}\left(\frac{x^2+1}{x}\right)}\,dx$$

$$= \alpha \log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right) + \beta\tan^{-1}\left(\frac{\gamma\left(x^2-1\right)}{x}\right) + \delta\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

Where C is an arbitrary constant, then the value of $10(\alpha+\beta\gamma+\delta)$ is equal to ....

Ans. **(6)**

Sol. $\int \frac{x^2-1}{\left(x^4+3x^2+1\right)\tan^{-1}\left(\frac{x^2+1}{x}\right)}\,dx + \int \frac{1}{x^4+3x^2+1}\,dx$

$$\int \frac{1-\frac{1}{x^2}}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}dx + \int \frac{dx}{x^4+3x^2+1}$$

$$\underset{I_1}{\downarrow} \qquad\qquad\qquad \underset{I_2}{\downarrow}$$

$$\tan^{-1}\left(x+\frac{1}{x}\right) = t$$

$$I_1 = \int \frac{dt}{t}$$

$$I_1 = \ln(t) = \ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right|$$

Now

$$I_2 = \int \frac{dx}{x^4+3x^2+1}$$

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+3x^2+1}dx$$

$$= \frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{x^2+3+\frac{1}{x^2}}dx - \int \frac{\left(1-\frac{1}{x^2}\right)}{x^2+3+\frac{1}{x^2}}dx\right]$$

$$\frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+5} - \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2+1}dx\right]$$

$$= \qquad \downarrow \qquad\qquad \downarrow$$

$$x-\frac{1}{x} = u \qquad\qquad x+\frac{1}{x} = v$$

$$= \frac{1}{2}\left[\int \frac{du}{u^2+\left(\sqrt{5}\right)^2} - \int \frac{dv}{v^2+1}\right]$$

$$I_2 = \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{5}}\right) - \frac{1}{2}\tan^{-1}\left(x+\frac{1}{x}\right)$$

$$I = I_1 + I_2 = \ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + \frac{1}{2\sqrt{5}}\ln\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$$\alpha = 1,\ \beta = \frac{1}{2\sqrt{5}},\ \lambda = \frac{1}{\sqrt{5}},\ \delta = -\frac{1}{2}$$

$$10(\alpha+\beta\lambda+\delta) = 10\left[1+\frac{1}{10}-\frac{1}{2}\right]$$

$$= 10\left(\frac{1}{10}+\frac{1}{2}\right)$$

$$= 1 + 5 = 6$$