Notation and Terminology

$\mathbb{N} =$ the set of all positive integers.

$\mathbb{Z} =$ the set of all integers.

$\mathbb{Q} =$ the set of all rational numbers.

$\mathbb{R} =$ the set of all real numbers.

$\mathbb{R}^n =$ the n-dimensional Euclidean space.

$\mathbb{C} =$ the set of all complex numbers.

$M_n(\mathbb{R})$ = the real vector space of all $n \times n$ matrices with entries in $\mathbb{R}$.

$M_n(\mathbb{C})$ = the complex vector space of all $n \times n$ matrices with entries in $\mathbb{C}$.

$\gcd(m, n)$ = the greatest common divisor of the integers m and n.

$M^{\mathsf{T}} =$ the transpose of the matrix M.

$A - B =$ the complement of the set B in the set A, that is, $\{x \in A : x \notin B\}$.

$\ln x =$ the natural logarithm of x (to the base e).

$|x| =$ the absolute value of x.

$y', y'', y''' =$ the first, second and the third derivatives of the function y, respectively.

$S_n =$ the symmetric group consisting of all permutations of $\{1,2, \dots, n\}$.

$\mathbb{Z}_n =$ the additive group of integers modulo n.

$f \circ g$ is the composite function defined by $(f \circ g)(x) = f(g(x))$.

The phrase '**real vector space**' refers to a vector space over $\mathbb{R}$.

Section A: Q.1 – Q.10 carry ONE mark each.	
Q.1	Consider the 2×2 matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$. If the eighth power of M satisfies $M^8 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the value of x is
(A)	21
(B)	22
(C)	34
(D)	35
Q.2	The rank of the 4×6 matrix $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ with entries in $\mathbb{R}$, is
(A)	1
(B)	2
(C)	3
(D)	4

Q.3	Let V be the real vector space consisting of all polynomials in one variable with real coefficients and having degree at most 6, together with the zero polynomial. Then which one of the following is true?
(A)	$\{f \in V : f(1/2) \notin \mathbb{Q}\}$ is a subspace of V.
(B)	$\{f \in V : f(1/2) = 1\}$ is a subspace of V.
(C)	$\{f \in V : f(1/2) = f(1)\}$ is a subspace of V.
(D)	$\{f \in V : f'(1/2) = 1\}$ is a subspace of V.
Q.4	Let G be a group of order 2022. Let H and K be subgroups of G of order 337 and 674, respectively. If $H \cup K$ is also a subgroup of G, then which one of the following is FALSE?
(A)	H is a normal subgroup of $H \cup K$.
(B)	The order of $H \cup K$ is 1011.
(C)	The order of $H \cup K$ is 674.
(D)	K is a normal subgroup of $H \cup K$.

Q.5	The radius of convergence of the power series $$\sum_{n=1}^{\infty}\left(\frac{n^3}{4^n}\right)x^{5n}$$ is
(A)	4
(B)	$\sqrt[5]{4}$
(C)	$\frac{1}{4}$
(D)	$\frac{1}{\sqrt[5]{4}}$

Q.6	Let (x_n) and (y_n) be sequences of real numbers defined by $$x_1 = 1, \quad y_1 = \frac{1}{2}, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n} \quad \text{for all } n \in \mathbb{N}.$$ Then which one of the following is true?
(A)	(x_n) is convergent, but (y_n) is not convergent.
(B)	(x_n) is not convergent, but (y_n) is convergent.
(C)	Both (x_n) and (y_n) are convergent and $\lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n$.
(D)	Both (x_n) and (y_n) are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.
Q.7	Suppose $$a_n = \frac{3^n + 3}{5^n - 5} \quad \text{and} \quad b_n = \frac{1}{(1+n^2)^{\frac{1}{4}}} \quad \text{for } n = 2,3,4,\ldots.$$ Then which one of the following is true?
(A)	Both $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ are convergent.
(B)	Both $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ are divergent.
(C)	$\sum_{n=2}^{\infty} a_n$ is convergent and $\sum_{n=2}^{\infty} b_n$ is divergent.
(D)	$\sum_{n=2}^{\infty} a_n$ is divergent and $\sum_{n=2}^{\infty} b_n$ is convergent.

Q.8	Consider the series $$\sum_{n=1}^{\infty} \frac{1}{n^m\left(1+\frac{1}{n^p}\right)}$$ where m and p are real numbers. Under which of the following conditions does the above series converge?
(A)	$m > 1$.
(B)	$0 < m < 1$ and $p > 1$.
(C)	$0 < m \leq 1$ and $0 \leq p \leq 1$.
(D)	$m = 1$ and $p > 1$.

Q.9	Let c be a positive real number and let $u: \mathbb{R}^2 \to \mathbb{R}$ be defined by $$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} e^{s^2} ds \quad \text{for} \quad (x,t) \in \mathbb{R}^2.$$ Then which one of the following is true?
(A)	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}^2$.
(B)	$\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}^2$.
(C)	$\dfrac{\partial u}{\partial t}\dfrac{\partial u}{\partial x} = 0$ on $\mathbb{R}^2$.
(D)	$\dfrac{\partial^2 u}{\partial t\, \partial x} = 0$ on $\mathbb{R}^2$.

Q.10	Let $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Consider the functions $$u : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R} \quad \text{and} \quad v : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$$ given by $$u(x,y) = x - \frac{x}{x^2+y^2} \quad \text{and} \quad v(x,y) = y + \frac{y}{x^2+y^2}.$$ The value of the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ at the point $(\cos\theta, \sin\theta)$ is equal to
(A)	$4\sin\theta$.
(B)	$4\cos\theta$.
(C)	$4\sin^2\theta$.
(D)	$4\cos^2\theta$.

Section A: Q.11 – Q.30 Carry TWO marks each.	
Q.11	Consider the open rectangle $G = \{(s,t) \in \mathbb{R}^2 : \ 0 < s < 1 \text{ and } 0 < t < 1\}$ and the map $T: G \to \mathbb{R}^2$ given by $$T(s,t) = \left(\frac{\pi s(1-t)}{2}, \ \frac{\pi(1-s)}{2}\right) \quad \text{for } (s,t) \in G.$$ Then the area of the image $T(G)$ of the map T is equal to
(A)	$\frac{\pi}{4}$
(B)	$\frac{\pi^2}{4}$
(C)	$\frac{\pi^2}{8}$
(D)	1

Q.12	Let T denote the sum of the convergent series $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$ and let S denote the sum of the convergent series $$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots=\sum_{n=1}^{\infty}a_n,$$ where $$a_{3m-2}=\frac{1}{2m-1},\quad a_{3m-1}=\frac{-1}{4m-2}\quad \text{and} \quad a_{3m}=\frac{-1}{4m}\quad \text{for } m\in\mathbb{N}.$$ Then which one of the following is true?
(A)	$T=S$ and $S\neq 0$.
(B)	$2T=S$ and $S\neq 0$.
(C)	$T=2S$ and $S\neq 0$.
(D)	$T=S=0$.

Q.13	Let $u: \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that $u(0) > 0$ and $u'(0) > 0$. Suppose u satisfies $$u''(x) = \frac{u(x)}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$ Consider the following two statements: I. The function uu' is monotonically increasing on $[0, \infty)$. II. The function u is monotonically increasing on $[0, \infty)$. Then which one of the following is correct?
(A)	Both I and II are false.
(B)	Both I and II are true.
(C)	I is false, but II is true.
(D)	I is true, but II is false.

Q.14	The value of $$\lim_{n\to\infty}\sum_{k=2}^{n}\frac{\sqrt{n+1}-\sqrt{n}}{k(\ln k)^2}$$ is equal to
(A)	∞
(B)	1
(C)	e
(D)	0

Q.15	For $t \in \mathbb{R}$, let $[t]$ denote the greatest integer less than or equal to t. Define functions $h: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ by $$h(x,y) = \begin{cases} \frac{-1}{x^2-y} & \text{if } x^2 \neq y, \\ 0 & \text{if } x^2 = y \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$ Then which one of the following is FALSE?
(A)	$\lim_{(x,y)\to(\sqrt{2},\pi)} \cos\left(\frac{x^2 y}{x^2+1}\right) = \frac{-1}{2}.$
(B)	$\lim_{(x,y)\to(\sqrt{2},2)} e^{h(x,y)} = 0.$
(C)	$\lim_{(x,y)\to(e,e)} \ln\left(x^{y-[y]}\right) = e - 2.$
(D)	$\lim_{(x,y)\to(0,0)} e^{2y} g(x) = 1.$

Q.16	Let $P \in M_4(\mathbb{R})$ be such that P^4 is the zero matrix, but P^3 is a nonzero matrix. Then which one of the following is FALSE?
(A)	For every nonzero vector $v \in \mathbb{R}^4$, the subset $\{v, Pv, P^2v, P^3v\}$ of the real vector space $\mathbb{R}^4$ is linearly independent.
(B)	The rank of P^k is $4 - k$ for every $k \in \{1,2,3,4\}$.
(C)	0 is an eigenvalue of P.
(D)	If $Q \in M_4(\mathbb{R})$ is such that Q^4 is the zero matrix, but Q^3 is a nonzero matrix, then there exists a nonsingular matrix $S \in M_4(\mathbb{R})$ such that $S^{-1}QS = P$.

Q.17	For $X, Y \in M_2(\mathbb{R})$, define $(X, Y) = XY - YX$. Let $\mathbf{0} \in M_2(\mathbb{R})$ denote the zero matrix. Consider the two statements: $P:\ \big(X,(Y,Z)\big) + \big(Y,(Z,X)\big) + \big(Z,(X,Y)\big) = \mathbf{0}$ for all $X, Y, Z \in M_2(\mathbb{R})$. $Q:\ \big(X,(Y,Z)\big) = \big((X,Y),Z\big)$ for all $X, Y, Z \in M_2(\mathbb{R})$. Then which one of the following is correct?
(A)	Both P and Q are true.
(B)	P is true, but Q is false.
(C)	P is false, but Q is true.
(D)	Both P and Q are false.

Q.18	Consider the system of linear equations $$\begin{aligned} x+y+t &= 4, \\ 2x-4t &= 7, \\ x+y+z &= 5, \\ x-3y-z-10t &= \lambda, \end{aligned}$$ where x, y, z, t are variables and λ is a constant. Then which one of the following is true?
(A)	If $\lambda = 1$, then the system has a unique solution.
(B)	If $\lambda = 2$, then the system has infinitely many solutions.
(C)	If $\lambda = 1$, then the system has infinitely many solutions.
(D)	If $\lambda = 2$, then the system has a unique solution.
Q.19	Consider the group $(\mathbb{Q}, +)$ and its subgroup $(\mathbb{Z}, +)$. For the quotient group $\mathbb{Q}/\mathbb{Z}$, which one of the following is FALSE?
(A)	$\mathbb{Q}/\mathbb{Z}$ contains a subgroup isomorphic to $(\mathbb{Z}, +)$.
(B)	There is exactly one group homomorphism from $\mathbb{Q}/\mathbb{Z}$ to $(\mathbb{Q}, +)$.
(C)	For all $n \in \mathbb{N}$, there exists $g \in \mathbb{Q}/\mathbb{Z}$ such that the order of g is n.
(D)	$\mathbb{Q}/\mathbb{Z}$ is not a cyclic group.

Q.20	For $P \in M_5(\mathbb{R})$ and $i, j \in \{1,2, \dots ,5\}$, let p_{ij} denote the $(i,j)^{\text{th}}$ entry of P. Let $$S = \{P \in M_5(\mathbb{R}) : p_{ij} = p_{rs} \text{ for } i,j,r,s \in \{1,2,\dots,5\} \text{ with } i + r = j + s\}.$$ Then which one of the following is FALSE?
(A)	S is a subspace of the vector space over $\mathbb{R}$ of all 5×5 symmetric matrices.
(B)	The dimension of S over $\mathbb{R}$ is 5.
(C)	The dimension of S over $\mathbb{R}$ is 11.
(D)	If $P \in S$ and all the entries of P are integers, then 5 divides the sum of all the diagonal entries of P.

Q.21	On the open interval $(-c, c)$, where c is a positive real number, $y(x)$ is an infinitely differentiable solution of the differential equation $$\frac{dy}{dx} = y^2 - 1 + \cos x,$$ with the initial condition $y(0) = 0$. Then which one of the following is correct?
(A)	$y(x)$ has a local maximum at the origin.
(B)	$y(x)$ has a local minimum at the origin.
(C)	$y(x)$ is strictly increasing on the open interval $(-\delta, \delta)$ for some positive real number δ.
(D)	$y(x)$ is strictly decreasing on the open interval $(-\delta, \delta)$ for some positive real number δ.

Q.22	Let $H : \mathbb{R} \to \mathbb{R}$ be the function given by $H(x) = \frac{1}{2}(e^x + e^{-x})$ for $x \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $$f(x) = \int_0^{\pi} H(x \sin\theta) d\theta \quad \text{for } x \in \mathbb{R}.$$ Then which one of the following is true?
(A)	$xf''(x) + f'(x) + xf(x) = 0$ for all $x \in \mathbb{R}$.
(B)	$xf''(x) - f'(x) + xf(x) = 0$ for all $x \in \mathbb{R}$.
(C)	$xf''(x) + f'(x) - xf(x) = 0$ for all $x \in \mathbb{R}$.
(D)	$xf''(x) - f'(x) - xf(x) = 0$ for all $x \in \mathbb{R}$.

Q.23	Consider the differential equation $$y'' + ay' + y = \sin x \quad \text{for } x \in \mathbb{R}. \qquad (**)$$ Then which one of the following is true?
(A)	If $a = 0$, then all the solutions of $(**)$ are unbounded over $\mathbb{R}$.
(B)	If $a = 1$, then all the solutions of $(**)$ are unbounded over $(0, \infty)$.
(C)	If $a = 1$, then all the solutions of $(**)$ tend to zero as $x \to \infty$.
(D)	If $a = 2$, then all the solutions of $(**)$ are bounded over $(-\infty, 0)$.
Q.24	For $g \in \mathbb{Z}$, let $\overline{g} \in \mathbb{Z}_{37}$ denote the residue class of g modulo 37. Consider the group $U_{37} = \{\overline{g} \in \mathbb{Z}_{37} : 1 \leq g \leq 37 \text{ with } \gcd(g, 37) = 1\}$ with respect to multiplication modulo 37. Then which one of the following is FALSE?
(A)	The set $\{\overline{g} \in U_{37} : \overline{g} = (\overline{g})^{-1}\}$ contains exactly 2 elements.
(B)	The order of the element $\overline{10}$ in U_{37} is 36.
(C)	There is exactly one group homomorphism from U_{37} to $(\mathbb{Z}, +)$.
(D)	There is exactly one group homomorphism from U_{37} to $(\mathbb{Q}, +)$.

Q.25	For some real number c with $0 < c < 1$, let $\phi: (1-c, 1+c) \to (0, \infty)$ be a differentiable function such that $\phi(1) = 1$ and $y = \phi(x)$ is a solution of the differential equation $$(x^2 + y^2)dx - 4xy\, dy = 0.$$ Then which one of the following is true?
(A)	$\left(3\left(\phi(x)\right)^2 + x^2\right)^2 = 4x.$
(B)	$\left(3\left(\phi(x)\right)^2 - x^2\right)^2 = 4x.$
(C)	$\left(3\left(\phi(x)\right)^2 + x^2\right)^2 = 4\phi(x).$
(D)	$\left(3\left(\phi(x)\right)^2 - x^2\right)^2 = 4\phi(x).$

Q.26	For a 4×4 matrix $M \in M_4(\mathbb{C})$, let $\bar{M}$ denote the matrix obtained from M by replacing each entry of M by its complex conjugate. Consider the real vector space $$H = \{M \in M_4(\mathbb{C}) : \quad M^\top = \bar{M}\}$$ where $M^\top$ denotes the transpose of M. The dimension of H as a vector space over $\mathbb{R}$ is equal to
(A)	6
(B)	16
(C)	15
(D)	12

Q.27	Let a, b be positive real numbers such that $a < b$. Given that $$\lim_{N\to\infty}\int_0^N e^{-t^2}dt = \frac{\sqrt{\pi}}{2},$$ the value of $$\lim_{N\to\infty}\int_0^N \frac{1}{t^2}\left(e^{-at^2} - e^{-bt^2}\right)dt$$ is equal to
(A)	$\sqrt{\pi}(\sqrt{a} - \sqrt{b})$.
(B)	$\sqrt{\pi}(\sqrt{a} + \sqrt{b})$.
(C)	$-\sqrt{\pi}(\sqrt{a} + \sqrt{b})$.
(D)	$\sqrt{\pi}(\sqrt{b} - \sqrt{a})$.

Q.28	For $-1 \leq x \leq 1$, if $f(x)$ is the sum of the convergent power series $$x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\cdots+\frac{x^n}{n^2}+\cdots$$ then $f\left(\frac{1}{2}\right)$ is equal to
(A)	$\displaystyle\int_0^{\frac{1}{2}} \frac{\ln(1-t)}{t}\,dt.$
(B)	$\displaystyle-\int_0^{\frac{1}{2}} \frac{\ln(1-t)}{t}\,dt.$
(C)	$\displaystyle\int_0^{\frac{1}{2}} \mathrm{t}\ln(1+t)\,dt.$
(D)	$\displaystyle\int_0^{\frac{1}{2}} \mathrm{t}\ln(1-t)\,dt.$

Q.29	For $n \in \mathbb{N}$ and $x \in [1, \infty)$, let $$f_n(x) = \int_0^\pi \left(x^2 + (\cos\theta)\sqrt{x^2-1}\right)^n d\theta.$$ Then which one of the following is true?
(A)	$f_n(x)$ is not a polynomial in x if n is odd and $n \geq 3$.
(B)	$f_n(x)$ is not a polynomial in x if n is even and $n \geq 4$.
(C)	$f_n(x)$ is a polynomial in x for all $n \in \mathbb{N}$.
(D)	$f_n(x)$ is not a polynomial in x for any $n \geq 3$.
Q.30	Let P be a 3×3 real matrix having eigenvalues $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = -1$. Further, $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ are eigenvectors of the matrix P corresponding to the eigenvalues λ_1, λ_2 and λ_3, respectively. Then the entry in the first row and the third column of P is
(A)	0
(B)	1
(C)	-1
(D)	2

Section B: Q.31 – Q.40 Carry TWO marks each.	
Q.31	Let $(-c, c)$ be the largest open interval in $\mathbb{R}$ (where c is either a positive real number or $c = \infty$) on which the solution $y(x)$ of the differential equation $$\frac{dy}{dx} = x^2 + y^2 + 1 \quad \text{with initial condition} \quad y(0) = 0$$ exists and is unique. Then which of the following is/are true?
(A)	$y(x)$ is an odd function on $(-c, c)$.
(B)	$y(x)$ is an even function on $(-c, c)$.
(C)	$\big(y(x)\big)^2$ has a local minimum at 0.
(D)	$\big(y(x)\big)^2$ has a local maximum at 0.

Q.32	Let S be the set of all continuous functions $f: [-1,1] \to \mathbb{R}$ satisfying the following three conditions: (i) f is infinitely differentiable on the open interval $(-1,1)$, (ii) the Taylor series $$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$ of f at 0 converges to $f(x)$ for each $x \in (-1,1)$, (iii) $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$. Then which of the following is/are true?
(A)	$f(0) = 0$ for every $f \in S$.
(B)	$f'\left(\frac{1}{2}\right) = 0$ for every $f \in S$.
(C)	There exists $f \in S$ such that $f'\left(\frac{1}{2}\right) \neq 0$.
(D)	There exists $f \in S$ such that $f(x) \neq 0$ for some $x \in [-1,1]$.

Q.33	Define $f:[0,1] \to [0,1]$ by $$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ for some } m, n \in \mathbb{N} \text{ with } m \leq n \text{ and } \gcd(m,n) = 1, \\ 0 & \text{if } x \in [0,1] \text{ is irrational.} \end{cases}$$ and define $g:[0,1] \to [0,1]$ by $$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1]. \end{cases}$$ Then which of the following is/are true?
(A)	f is Riemann integrable on $[0,1]$.
(B)	g is Riemann integrable on $[0,1]$.
(C)	The composite function $f \circ g$ is Riemann integrable on $[0,1]$.
(D)	The composite function $g \circ f$ is Riemann integrable on $[0,1]$.

Q.34	Let S be the set of all functions $f: \mathbb{R} \to \mathbb{R}$ satisfying $$\lvert f(x) - f(y) \rvert^2 \leq \lvert x - y \rvert^3 \quad \text{for all } x, y \in \mathbb{R}.$$ Then which of the following is/are true?
(A)	Every function in S is differentiable.
(B)	There exists a function $f \in S$ such that f is differentiable, but f is not twice differentiable.
(C)	There exists a function $f \in S$ such that f is twice differentiable, but f is not thrice differentiable.
(D)	Every function in S is infinitely differentiable.

Q.35	A real-valued function $y(x)$ defined on $\mathbb{R}$ is said to be **periodic** if there exists a real number $T > 0$ such that $y(x+T) = y(x)$ for all $x \in \mathbb{R}$. Consider the differential equation $$\frac{d^2y}{dx^2} + 4y = \sin(ax), \quad x \in \mathbb{R}, \qquad (*)$$ where $a \in \mathbb{R}$ is a constant. Then which of the following is/are true?
(A)	All solutions of $(*)$ are periodic for every choice of a.
(B)	All solutions of $(*)$ are periodic for every choice of $a \in \mathbb{R} - \{-2, 2\}$.
(C)	All solutions of $(*)$ are periodic for every choice of $a \in \mathbb{Q} - \{-2, 2\}$.
(D)	If $a \in \mathbb{R} - \mathbb{Q}$, then there is a unique periodic solution of $(*)$.

Q.36	Let M be a positive real number and let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying $$\sqrt{u(x,y)^2 + v(x,y)^2} \geq M\sqrt{x^2+y^2} \quad \text{for all } (x,y) \in \mathbb{R}^2.$$ Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $$F(x,y) = \big(u(x,y), v(x,y)\big) \quad \text{for } (x,y) \in \mathbb{R}^2.$$ Then which of the following is/are true?
(A)	F is injective.
(B)	If K is open in $\mathbb{R}^2$, then $F(K)$ is open in $\mathbb{R}^2$.
(C)	If K is closed in $\mathbb{R}^2$, then $F(K)$ is closed in $\mathbb{R}^2$.
(D)	If E is closed and bounded in $\mathbb{R}^2$, then $F^{-1}(E)$ is closed and bounded in $\mathbb{R}^2$.

Q.37	Let G be a finite group of order at least two and let e denote the identity element of G. Let $\sigma: G \to G$ be a bijective group homomorphism that satisfies the following two conditions: (i) If $\sigma(g) = g$ for some $g \in G$, then $g = e$, (ii) $(\sigma \circ \sigma)(g) = g$ for all $g \in G$. Then which of the following is/are correct?
(A)	For each $g \in G$, there exists $h \in G$ such that $h^{-1}\sigma(h) = g$.
(B)	There exists $x \in G$ such that $x\sigma(x) \neq e$.
(C)	The map σ satisfies $\sigma(x) = x^{-1}$ for every $x \in G$.
(D)	The order of the group G is an odd number.

Q.38	Let (x_n) be a sequence of real numbers. Consider the set $$P = \{n \in \mathbb{N} : x_n > x_m \text{ for all } m \in \mathbb{N} \text{ with } m > n\}.$$ Then which of the following is/are true?
(A)	If P is finite, then (x_n) has a monotonically increasing subsequence.
(B)	If P is finite, then no subsequence of (x_n) is monotonically increasing.
(C)	If P is infinite, then (x_n) has a monotonically decreasing subsequence.
(D)	If P is infinite, then no subsequence of (x_n) is monotonically decreasing.
Q.39	Let V be the real vector space consisting of all polynomials in one variable with real coefficients and having degree at most 5, together with the zero polynomial. Let $T: V \to \mathbb{R}$ be the linear map defined by $T(1) = 1$ and $$T\big(x(x-1)\cdots(x-k+1)\big) = 1 \quad \text{for } 1 \le k \le 5.$$ Then which of the following is/are true?
(A)	$T(x^4) = 15$.
(B)	$T(x^3) = 5$.
(C)	$T(x^4) = 14$.
(D)	$T(x^3) = 3$.

Q.40	Let P be a fixed 3×3 matrix with entries in $\mathbb{R}$. Which of the following maps from $M_3(\mathbb{R})$ to $M_3(\mathbb{R})$ is/are linear?
(A)	$T_1: M_3(\mathbb{R}) \to M_3(\mathbb{R})$ given by $T_1(M) = MP - PM$ for $M \in M_3(\mathbb{R})$.
(B)	$T_2: M_3(\mathbb{R}) \to M_3(\mathbb{R})$ given by $T_2(M) = M^2P - P^2M$ for $M \in M_3(\mathbb{R})$.
(C)	$T_3: M_3(\mathbb{R}) \to M_3(\mathbb{R})$ given by $T_3(M) = MP^2 + P^2M$ for $M \in M_3(\mathbb{R})$.
(D)	$T_4: M_3(\mathbb{R}) \to M_3(\mathbb{R})$ given by $T_4(M) = MP^2 - PM^2$ for $M \in M_3(\mathbb{R})$.

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41	The value of the limit $$\lim_{n\to\infty}\left(\frac{(1^4+2^4+\cdots+n^4)}{n^5}+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{4n}}\right)\right)$$ is equal to __________. (*Rounded off to two decimal places*)

Q.42	Consider the function $u: \mathbb{R}^3 \to \mathbb{R}$ given by $$u(x_1, x_2, x_3) = x_1 x_2^4 x_3^2 - x_1^3 x_3^4 - 26 x_1^2 x_2^2 x_3^3.$$ Let $c \in \mathbb{R}$ and $k \in \mathbb{N}$ be such that $$x_1 \frac{\partial u}{\partial x_2} + 2x_2 \frac{\partial u}{\partial x_3}$$ evaluated at the point (t, t^2, t^3), equals ct^k for every $t \in \mathbb{R}$. Then the value of k is equal to ___________.
Q.43	Let $y(x)$ be the solution of the differential equation $$\frac{dy}{dx} + 3x^2 y = x^2, \qquad \text{for } x \in \mathbb{R},$$ satisfying the initial condition $y(0) = 4$. Then $\lim_{x\to\infty} y(x)$ is equal to ___________. (*Rounded off to two decimal places*)
Q.44	The sum of the series $$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$$ is equal to ___________. (*Rounded off to two decimal places*)
Q.45	The number of distinct subgroups of $\mathbb{Z}_{999}$ is ___________.

Q.46	The number of elements of order 12 in the symmetric group S_7 is equal to __________.
Q.47	Let $y(x)$ be the solution of the differential equation $$xy^2y' + y^3 = \frac{\sin x}{x} \quad \text{for } x > 0,$$ satisfying $y\left(\frac{\pi}{2}\right) = 0$. Then the value of $y\left(\frac{5\pi}{2}\right)$ is equal to __________. (*Rounded off to two decimal places*)
Q.48	Consider the region $$G = \{(x, y, z) \in \mathbb{R}^3 : 0 < z < x^2 - y^2,\ x^2 + y^2 < 1\}.$$ Then the volume of G is equal to __________. (*Rounded off to two decimal places*)

Q.49 Given that $y(x)$ is a solution of the differential equation

$$x^2 y'' + xy' - 4y = x^2$$

on the interval $(0, \infty)$ such that $\lim_{x \to 0^+} y(x)$ exists and $y(1) = 1$. The value of $y'(1)$ is equal to __________. (*Rounded off to two decimal places*)

Q.50 Consider the family $\mathcal{F}_1$ of curves lying in the region

$$\{(x, y) \in \mathbb{R}^2 : \ y > 0 \text{ and } 0 < x < \pi\}$$

and given by

$$y = \frac{c(1 - \cos x)}{\sin x}, \quad \text{where } c \text{ is a positive real number.}$$

Let $\mathcal{F}_2$ be the family of orthogonal trajectories to $\mathcal{F}_1$. Consider the curve $\mathcal{C}$ belonging to the family $\mathcal{F}_2$ passing through the point $\left(\frac{\pi}{3}, 1\right)$. If a is a real number such that $\left(\frac{\pi}{4}, a\right)$ lies on $\mathcal{C}$, then the value of a^4 is equal to __________. (*Rounded off to two decimal places*)

Section C: Q.51 – Q.60 Carry TWO marks each.	
Q.51	For $t \in \mathbb{R}$, let $[t]$ denote the greatest integer less than or equal to t. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be defined by $f(0,0) = g(0,0) = 0$ and $$f(x,y) = [x^2 + y^2]\frac{x^2y^2}{x^4 + y^4}, \qquad g(x,y) = [y^2]\frac{xy}{x^2 + y^2}$$ for $(x, y) \neq (0,0)$. Let E be the set of points of D at which both f and g are discontinuous. The number of elements in the set E is _________.
Q.52	If G is the region in $\mathbb{R}^2$ given by $$G = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1,\ \frac{x}{\sqrt{3}} < y < \sqrt{3}x,\ x > 0,\ y > 0\right\}$$ then the value of $$\frac{200}{\pi}\iint_G x^2 dx\, dy$$ is equal to ___________. (*Rounded off to two decimal places*)

Q.53	Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$ and let $A^{\top}$ denote the transpose of A. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be column vectors with entries in $\mathbb{R}$ such that $u_1^2 + u_2^2 = 1$ and $v_1^2 + v_2^2 + v_3^2 = 1$. Suppose $$Au = \sqrt{2}\, v \quad \text{and} \quad A^{\top} v = \sqrt{2}\, u.$$ Then $\|u_1 + 2\sqrt{2}\, v_1\|$ is equal to __________. (*Rounded off to two decimal places*)
Q.54	Let $f\colon [0, \pi] \to \mathbb{R}$ be the function defined by $$f(x) = \begin{cases} (x-\pi)e^{\sin x} & \text{if } 0 \le x \le \dfrac{\pi}{2}, \\ xe^{\sin x} + \dfrac{4}{\pi} & \text{if } \dfrac{\pi}{2} < x \le \pi. \end{cases}$$ Then the value of $$\int_0^{\pi} f(x)dx$$ is equal to __________. (*Rounded off to two decimal places*)

Q.55	Let r be the radius of convergence of the power series $$\frac{1}{3}+\frac{x}{5}+\frac{x^2}{3^2}+\frac{x^3}{5^2}+\frac{x^4}{3^3}+\frac{x^5}{5^3}+\frac{x^6}{3^4}+\frac{x^7}{5^4}+\cdots.$$ Then the value of r^2 is equal to ___________. (*Rounded off to two decimal places*)
Q.56	Define $f:\mathbb{R}^2\to\mathbb{R}$ by $$f(x,y)=x^2+2y^2-x \quad \text{for } (x,y)\in\mathbb{R}^2.$$ Let $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$ and $E=\left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{4}+\frac{y^2}{9}\le 1\right\}$. Consider the sets $$D_{\max}=\{(a,b)\in D : f \text{ has absolute maximum on } D \text{ at } (a,b)\},$$ $$D_{\min}=\{(a,b)\in D : f \text{ has absolute minimum on } D \text{ at } (a,b)\},$$ $$E_{\max}=\{(c,d)\in E : f \text{ has absolute maximum on } E \text{ at } (c,d)\},$$ $$E_{\min}=\{(c,d)\in E : f \text{ has absolute minimum on } E \text{ at } (c,d)\}.$$ Then the total number of elements in the set $$D_{\max}\cup D_{\min}\cup E_{\max}\cup E_{\min}$$ is equal to ___________.

Q.57	Consider the 4×4 matrix $M = \begin{pmatrix} 11 & 10 & 10 & 10 \\ 10 & 11 & 10 & 10 \\ 10 & 10 & 11 & 10 \\ 10 & 10 & 10 & 11 \end{pmatrix}$. Then the value of the determinant of M is equal to _________.
Q.58	Let σ be the permutation in the symmetric group S_5 given by $\sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 1, \quad \sigma(4) = 5, \quad \sigma(5) = 4.$ Define $$N(\sigma) = \{\tau \in S_5 : \sigma \circ \tau = \tau \circ \sigma\}.$$ Then the number of elements in $N(\sigma)$ is equal to __________.
Q.59	Let $f: (-1, 1) \to \mathbb{R}$ and $g: (-1, 1) \to \mathbb{R}$ be thrice continuously differentiable functions such that $f(x) \neq g(x)$ for every nonzero $x \in (-1, 1)$. Suppose $f(0) = \ln 2, \quad f'(0) = \pi, \quad f''(0) = \pi^2, \quad \text{and} \quad f'''(0) = \pi^9$ and $g(0) = \ln 2, \quad g'(0) = \pi, \quad g''(0) = \pi^2, \quad \text{and} \quad g'''(0) = \pi^3.$ Then the value of the limit $$\lim_{x \to 0} \frac{e^{f(x)} - e^{g(x)}}{f(x) - g(x)}$$ is equal to _________. (*Rounded off to two decimal places*)

Q.60	If $f: [0, \infty) \to \mathbb{R}$ and $g: [0, \infty) \to [0, \infty)$ are continuous functions such that $\int_0^{x^3+x^2} f(t)dt = x^2$ and $\int_0^{g(x)} t^2 dt = 9(x+1)^3$ for all $x \in [0, \infty)$, then the value of $$f(2) + g(2) + 16\, f(12)$$ is equal to __________. (*Rounded off to two decimal places*)