Multiple Choice Type Questions:

Q1. **A function $f: R \rightarrow$ R defined as $f(x) = x^2 - 4x + 5$ is :**

(a) injective but not surjective
(b) surjective but not injective
(c) both injective and surjective
(d) neither injective nor surjective

S1. **Ans.(d)**

Sol. Given, $f(x) = x^2 - 4x + 5$
Here $f(0) = f(4) = 5$
Hence, $f(x)$ is not one-one.
To check whether the function is onto or not, we have to find range of function.
Let $y = x^2 - 4x + 5 \Rightarrow x^2 - 4x + 5 - y = 0$
$\therefore D = (4)^2 - 4(1)(5 - y) \geq 0 \ \forall x \in R$
$\Rightarrow 16 - 20 + 4y \geq 0 \Rightarrow 4y - 4 \geq 0$
$\Rightarrow 4(y - 1) \geq 0 \Rightarrow y \geq 1$
Here, range $= (1, \infty)$
Here, Co-domain $\neq$ Range
So, $f(x)$ is not onto.
{$\because$ If for a function co-domain $\neq$ range, then the function is not onto.}

Q2. If $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0$, **then the value of α is**

(a) 1
(b) 2
(c) 3
(d) 4

S2. **Ans.(d)**

Sol. $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0 \Rightarrow \alpha(2 - 4) - 3(1 - 1) + 4(4 - 2) = 0$
$\Rightarrow -2\alpha + 8 = 0 \Rightarrow 2\alpha = 8 \Rightarrow \alpha = 4$

Q3. **Derivative of $e^{\sin^2 x}$ wirth respect to cos is**

(a) $\sin x e^{\sin^2 x}$
(b) $\cos x e^{\sin^2 x}$
(c) $-2\cos x e^{\sin^2 x}$
(d) $-2\sin^2 x \cos x e^{\sin^2 x}$

S3. **(c)**

Sol. let $P = e^{\sin^2 x}$ and $Q = \cos x$
Differentiating both sides w.r.t. x we get
$\frac{dP}{dx} = e^{\sin^2 x}.\, 2\sin x \,.\cos x$ and $\frac{dQ}{dx} = -\sin x$
Now, $\frac{dP}{dQ} = \frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \frac{2e^{\sin^2 x}\sin x \cos x}{-\sin x} = -2e^{\sin^2 x}\cos x$

Q4. **If $\int_{-2}^{3} x^2 dx = k\int_{0}^{2} x^2 dx + \int_{2}^{3} x^2\, dx$, then the value of k is**

(a) 2
(b) 1
(c) 0
(d) $\frac{1}{2}$

S4. Ans.(a)

Sol. Given, $\int_{-2}^{3} x^2 dx = k\int_{0}^{2} x^2 dx + \int_{2}^{3} x^2 dx$

$\Rightarrow \left[\frac{x^3}{3}\right]_{-2}^{3} = k\left[\frac{x^3}{3}\right]_{0}^{2} + \left[\frac{x^3}{3}\right]_{2}^{3}$

$\Rightarrow \frac{27}{3} + \frac{8}{3} = k\left(\frac{8}{3}\right) + \frac{27}{3} - \frac{8}{3} \Rightarrow \frac{8k}{3} = \frac{16}{3} \Rightarrow k = 2$

Q5. The integrating factor of the differential equation $(x + 3y^2)\frac{dy}{dx} = y$ is

(a) y
(b) $-y$
(c) $\frac{1}{y}$
(d) $-\frac{1}{y}$

S5. Ans. (c)

Sol. We have, $(x + 3y^2)\frac{dy}{dx} = y$

$\Rightarrow \frac{x+3y^2}{y} = \frac{dx}{dy} \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$

This is a linear differential equation.

$\therefore$ I.F. $= e^{-\int\frac{dy}{y}} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$

Very Short Answer Type Questions:

Q6. The value of λ for which the vectors $2\hat{\imath} - \lambda\hat{\jmath} + \hat{k}$ and $\hat{\imath} + 2\hat{\jmath} - \hat{k}$ are orthogonal is ______.

S6.

Sol. Let $\vec{a} = 2\hat{\imath} - \lambda\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

We know, $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow (2\hat{\imath} - \lambda\hat{\jmath} + \hat{k})\cdot(\hat{\imath} + 2\hat{\jmath} - \hat{k}) = 0$

$\Rightarrow 2 - 2\lambda - 1 = 0 \Rightarrow 1 - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$.

Q7. The equations of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line and find the coordinates of a point through which it passes.

S7.

Sol. The given line is $5x - 3 = 15y + 7 = 3 - 10z$

$\Rightarrow \frac{x-\frac{3}{5}}{\frac{1}{5}} = \frac{y+\frac{7}{15}}{\frac{1}{15}} = \frac{z-\frac{3}{10}}{-\frac{1}{10}}$

Its direction ratios are $\frac{1}{5}, \frac{1}{15}, \frac{1}{10}$

i.e., Its direction ratios are proportional to $6, 2, -3$.

Now, $\sqrt{6^2 + 2^2 + (-3)^2} = 7$

$\therefore$ Its direction cosines are $\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$.

Q8. Check if the relation R in the set $\mathbb{R}$ of real numbers defined as $R = \{(a,b): a < b\}$ is (i) symmetric, (ii) transitive

S8.

Sol. We have, $R = \{(a,b): a < b\}$, where $a, b \in \mathbb{R}$

(i) Symmetric: Let $(x,y) \in R, i.e., xRy$.

$\Rightarrow x < y$

But $y \nless x$, so$(x,y) \in R \Rightarrow (y,x) \notin R$

Thus, R is not symmetric.

(ii) Transitive: Let $(x,y), (y,z) \in R$

$\Rightarrow x < y$ and $y < z \Rightarrow x < z$

$\Rightarrow (x,z) \in R$. Thus, R is transitive.

Q9. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$

S9.

Sol. The given differential equation is

$x^2 \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^4$ $\therefore$ Its order is 2 and degree is 1.

Q10. Find the domain of the function $f(x) = \sin^{-1}(x^2 - 4)$. Also, find its range.

S10.

Sol. For domain, $-1 \le x^2 - 4 \le 1$

$\Rightarrow 3 \le x^2 \le 5 \Rightarrow \sqrt{3} \le |x| \le \sqrt{5}$

$\Rightarrow x \in [-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$

Let $y = \sin^{-1}(x^2 - 4) \Rightarrow \sin y + 4 = x^2$

Now, $3 \le x^2 \le 5 \Rightarrow 3 \le \sin y + 4 \le 5$

$\Rightarrow -1 \le \sin y \le 1$

$\Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2} \therefore R_f = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Q11. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$.

S11.

Sol. Given, $A^2 = I$

$\therefore$ The simplified value of $(A - I)^3 + (A + I)^3 - 7A$

$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$

$= 2A^3 + 6AI^2 - 7A = 2AA^2 + 6AI - 7A$

$= 2AI + 6A - 7A = 2A - A = A$

$\{\because I \cdot A = A \cdot I = A$ and $A^2 = I\}$

Q12. Show that $\mathrm{f(x) = e^x - e^{-x} + x - \tan^{-1} x}$ is strictly increasing in its domain.

S12.

Sol. We have $\mathrm{f(x) = e^x - e^{-x} + x - \tan^{-1} x}$

$\Rightarrow \mathrm{f'(x) = e^x + e^{-x} + 1 - \frac{1}{1+x^2}}$

$\mathrm{= e^x + \frac{1}{e^x} + \frac{1+x^2-1}{1+x^2} = e^x + \frac{1}{e^x} + \frac{x^2}{1+x^2} > 0}$ for all $\mathrm{x \in D_f}$

$(\because \mathrm{e^x > 0}$ for all x)

So, $f(x)$ is strictly increasing in its domain.

Q13. Find : $\int \cos^3 x\, e^{\log \sin x}\, dx$

S13.

Sol. Let $I = \int \cos^3 x\, e^{\log \sin x} dx \Rightarrow I = \int \cos^3 x \sin x\, dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$ or

$\sin x\, dx = -dt$

$\therefore I = -\int t^3\, dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$

Q14. Find the general solution of the differential equation $\mathrm{e^{y-x} \frac{dy}{dx} = 1}$.

S14.

Sol. We have, $e^{y-x} \frac{dy}{dx} = 1 \Rightarrow e^y \cdot e^{-x} \frac{dy}{dx} = 1$

$\Rightarrow e^y dy = e^x dx$

Integrating both sides we get,

$e^y = e^x + c, y = \log(e^x + c)$

Q15. $\vec{a}$ and $\vec{b}$ are two −unit vectors such that $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$.

S15.

Sol. Given $\vec{a}$ and $\vec{b}$ are unit vectors

$\therefore |\vec{a}| = |\vec{b}| = 1$ ……. (i)

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Also, $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$ (Given)

$\Rightarrow |2\vec{a} + 3\vec{b}|^2 = |3\vec{a} - 2\vec{b}|^2$

$\Rightarrow (2\vec{a}+3\vec{b})\cdot(2\vec{a}+3\vec{b}) = (3\vec{a}-2\vec{b})\cdot(3\vec{a}-2\vec{b})$
$\Rightarrow 4(\vec{a}\cdot\vec{a}) + 6(\vec{a}\cdot\vec{b}) + 6(\vec{b}\cdot\vec{a}) + 9(\vec{b}\cdot\vec{b})$
$= 9(\vec{a}\cdot\vec{a}) - 6(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{a}) + 4(\vec{b}\cdot\vec{b})$
$\Rightarrow 4|\vec{a}|^2 + 12(\vec{a}\cdot\vec{b}) + 9|\vec{b}|^2 = 9|\vec{a}|^2 - 12(\vec{a}\cdot\vec{b}) + 4|\vec{b}|^2$
$\Rightarrow 5|\vec{a}|^2 - 5|\vec{b}|^2 - 24|\vec{a}||\vec{b}|\cos\theta = 0$
$\Rightarrow 5\cdot 1 - 5\cdot 1 - 24\,cos\theta = 0$
$(\because |\vec{a}| = |\vec{b}| = 1)$
$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$.

Short Answer Type Questions:

Q16. **Find : $\int \frac{1}{x[(\log x)^2 - 3\log x - 4]} dx$**

S16.

Sol. Let $I = \int \frac{1}{x[(\log x)^2 - 3\log x - 4]} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore I = \int \frac{dt}{t^2-3t-4} \Rightarrow I = \int \frac{dt}{(t+1)(t-4)}$ ...(i)

Using partial fraction, we have $\frac{1}{(t+1)(t-4)} = \frac{A}{t+1} + \frac{B}{t-4}$

$\Rightarrow 1 = A(t-4) + B(t+1)$

On comparing terms of same coefficients, we get

$A + B = 0$ ...(ii)

$-4A + B = 1$ ...(iii)

Solving (ii) and (iii), we get $A = \frac{-1}{5}, B = \frac{1}{5}$

$\therefore I = \int \frac{-1}{5(t+1)} dt + \int \frac{1}{5(t-4)} dt$ (Using(i))

$= -\frac{1}{5}\log|t+1| + \frac{1}{5}\log|t-4| + C$

$= \frac{-1}{5}\log|\log x + 1| + \frac{1}{5}\log|\log x - 4| + C$

$\Rightarrow I = \frac{1}{5}\log\left|\frac{\log x - 4}{\log x + 1}\right| + C$

Q17. **Find the particular solution of the differential equation, $\frac{dy}{dx} - 2xy = 3x^2e^{x^2}; y(0) = 5$.**

S17.

Sol. Given, $\frac{dy}{dx} - 2xy = 3x^2e^{x^2}$

Which is in the form of $\frac{dy}{dx} + Py = Q$,

Where $P = -2x, Q = 3x^2e^{x^2}$

$\therefore$ I.F. $= e^{\int Pdx} = e^{\int(-2x)dx} = e^{-x^2}$

$\therefore$ Required solution of given differential equation is

$y \times$ I.F. $= \int(Q \times I.F)dx + C$

$= \int(3x^2e^{x^2})\,e^{-x^2}dx + C$

$\Rightarrow y(e^{-x^2}) = 3\int x^2\,dx + C = x^3 + C$

When $x = 0$, then $y = 5 \therefore 5(e^{-0}) = 0 + C$

$\Rightarrow C = 5$

$\therefore$ Particular solution, $y(e^{-x^2}) = x^3 + 5$

Q18. **If a line makes an angle α, β, γ with the coordinate axes, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.**

S18.

Sol. Here, the direction cosines of the given line are $\cos\alpha, \cos\beta, \cos\gamma$ and $\cos^2\alpha\cos\beta, \cos\gamma = 1$

By using $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

$\cos^2\beta = \frac{1+\cos 2\beta}{2}$ and so on.

$\Rightarrow \frac{1}{2}[\cos 2\alpha + 1 + \cos 2\beta + 1 + \cos 2\gamma + 1]$

$= 1.$

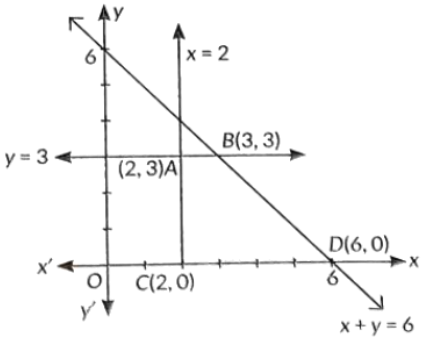
$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1.$

Q19. Solve the following linear programming problem graphically:

Maximise $z = 2x + 3y$
Subject to the constraints
$x + y \leq 6$
$x \geq 2$
$y \leq 3$
$x \geq 0, y \geq 0$

S19.

Sol. Consider the graph given below:



Corner points are A(2,3), B(3,3), C(2,0) and D(6,0)

$Z_A = 4 + 9 = 13$

$Z_B = 6 + 9 = 15$

$Z_c = 4$

$Z_D = 12$

Maximum value of z is 15 at (3,3).

Q20. The probability that A hits the targets is $\frac{1}{3}$ and the probability that B hits it, is $\frac{2}{5}$. If both try to hit the target independently, find the probability that the target is hit.

S20.

Sol. Let P(A) = Probability that A hits the target = $\frac{1}{3}$

Let P(B) = Probability that B hits the target = $\frac{2}{5}$

P(hits the target) = P(atleast one of A and B hits target)

= 1 – P(none hits)

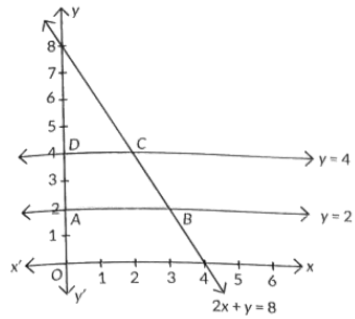
$= 1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{5}\right)$

$= 1 - \frac{2}{3} \times \frac{3}{5} = \frac{3}{5}$

Q21. Sketch the region bounded by the lines $2x + y = 8$, $y = 2$, $y = 4$ and y-axis. Hence, obtain its area using integration.

S21.

Sol. From the graph ABCD is the required region.



Now area=

$$\int_2^4 \left(\frac{8-y}{2}\right)dy = \frac{1}{2}\int_2^4 (8-y)dy$$

$$= \frac{1}{2}\left[8y - \frac{y^2}{2}\right]_2^4 = \frac{1}{2}\left[\left(32 - \frac{16}{2}\right) - \left(16 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{2} \times 10 = 5 \text{ sq. units}$$

Q22. If $y = \left(x + \sqrt{x^2 - 1}\right)^2$, then show that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$

S22.

Sol. We have $y = \left(x + \sqrt{x^2 - 1}\right)^2$

Now, $\frac{dy}{dx} = 2(x + \sqrt{x^2 - 1})\left(1 + \frac{2x}{2\sqrt{x^2-1}}\right)$

$= 2\left(x + \sqrt{x^2 - 1}\right)\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$

$= \frac{2\left(x+\sqrt{x^2-1}\right)^2}{\sqrt{x^2-1}}$

$\therefore \sqrt{x^2 - 1}\frac{dy}{dx} = 2y$

Squaring both sides, we get

$(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$

Hence proved

Q23. Simplify : $\cos^{-1} x + \cos^{-1}\left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right]; \frac{1}{2} \le x \le 1$

S23.

Sol. $f(x) = cos^{-1}x + cos^{-1}\left[\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right]$

$= \cos^{-1}x + \cos^{-1}\left[\frac{x}{2} + \frac{1}{2}\sqrt{3(1 - x^2)}\right]$

Let $\cos\theta = x \Rightarrow \theta = \cos^{-1} x$

$\therefore f(x) = \cos^{-1}(\cos\theta) + \cos^{-1}\left[\frac{\cos\theta}{2} + \frac{\sqrt{3}}{2}\sqrt{(1 - \cos^2\theta)}\right]$

$\Rightarrow f(x) = \theta + \cos^{-1}\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$

$\Rightarrow f(x) = \theta + \cos^{-1}\left[\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right]$

$\Rightarrow f(x) = \theta + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \theta\right)\right]$

$\{\because \cos(A - B) = \cos A \cos B + \sin A \sin B\}$

$\Rightarrow f(x) = \theta + \frac{\pi}{3} - \theta \quad \Rightarrow f(x) = \frac{\pi}{3}$

Q24. If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}(x \neq 0)$ respectively, find the values of $(M - m)$.

S24.

Sol. We have, $f(x) = x + \frac{1}{2}$

$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$

For maxima/minima, put $f'(x) = 0$

$$\Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow \frac{1}{x^2} = 1$$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

At $x = 1, f''(x) = \frac{2}{1} = 2 > 0$ (Minimum)

At $x = -1, f''(x) = \frac{2}{-1} < 0$ (Maximum)

$\therefore M = -1 + \frac{1}{(-1)} = -2$ and $m = 1 + \frac{1}{1} = 2$

So, $M - m = -2 - 2 = -4$

Q25. Solve graphically the following linear programming problem:
Maximise $z = 6x + 3y$,
Subject to the constraints
$4x + y \geq 80$,
$3x + 2y \leq 150$,
$x + 5y \geq 115$,
$x \geq 0, y \geq 0$.

S25.

Sol. We have, $z = 6x + 3y$

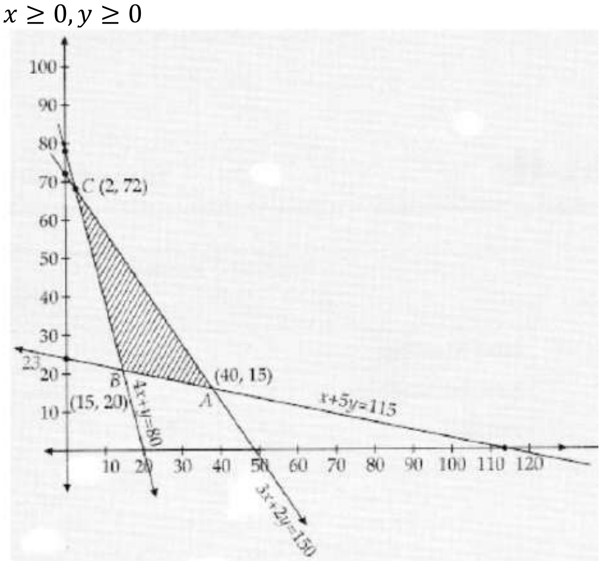
Subject to constraints

$4x + y \geq 80$

$3x + 2y \leq 150$

$x + 5y \geq 115$

$x \geq 0, y \geq 0$



$\therefore$ The corner points are (2, 72), (15, 20) and (40,15)

Corner points	$z = 6x + 3y$
(2, 72)	$z = 6 \times 2 + 3 \times 72$ $= 228$
(15,20)	$z = 6 \times 15 + 3 \times 20$ $= 150$
(40,15)	$z = 6 \times 40 + 3 \times 15$ $= 228$

$\therefore$ The maximum value of $z = 6x + 3y$ is 285 at the corner point (40,15).

Long Answer Type Questions:

Q26. Find the product of the matrices

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{3} & \mathbf{2} \\ \mathbf{3} & \mathbf{-3} & \mathbf{-4} \end{bmatrix}\begin{bmatrix} \mathbf{-6} & \mathbf{17} & \mathbf{13} \\ \mathbf{14} & \mathbf{5} & \mathbf{-8} \\ \mathbf{-15} & \mathbf{9} & \mathbf{-1} \end{bmatrix}$$ **and hence solve the system of linear equations.**

$\boldsymbol{x + 2y - 3z = -4; 2x + 3y + 2z = 2; 3x - 3y - 4z = 11}$

S26.

Sol. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, B = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$

$= \begin{bmatrix} -6 + 28 + 45 & 17 + 10 - 27 & 13 - 16 + 3 \\ -12 + 42 - 30 & 34 + 15 + 18 & 26 - 24 - 2 \\ -18 - 42 + 60 & 51 - 15 - 36 & 39 + 24 + 4 \end{bmatrix}$

$= \begin{bmatrix} 67 & 0 & 0 \\ 0 & 67 & 0 \\ 0 & 0 & 67 \end{bmatrix}$

$= 67\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 67I$ ...(i)

$\Rightarrow A\left(\frac{1}{67}B\right) = I \Rightarrow A^{-1} = \frac{1}{67}B$

$\Rightarrow A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$ ...(ii)

Given, the system of linear equations :

$x + 2y - 3z = -4$; $2x + 3y + 2z = 2$; and $3x - 3y - 4z = 11$

The system of equations can be written in the form $AX = B$, where

$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

Now, $X = A^{-1}B = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 24 + 34 + 143 \\ -56 + 10 - 88 \\ 60 + 18 - 11 \end{bmatrix}$

$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$\therefore x = 3, y = -2, z = 1$

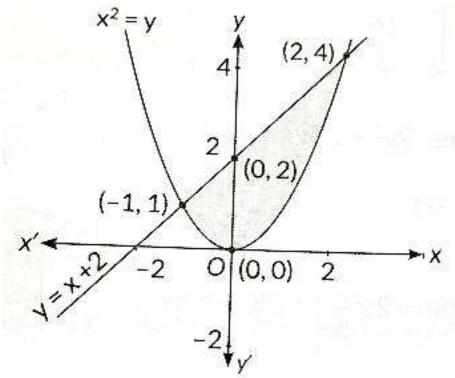
Q27. Find the area of the region bounded by the curves $\mathbf{x^2 = y, y = x + 2}$ and x-axis using integration.

S27.

Sol. The given curve is $x^2 = y \ldots\ldots(i)$

And the lines is $y = x + 2 \ldots\ldots\ldots(ii)$

Putting the value of y from eq. (i) in eq. (ii)



We get,
$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$
$\Rightarrow x^2 - 2x + x - 2 = 0$
$\Rightarrow x(x - 2) + (x - 2) = 0$
$\Rightarrow x = 2$ or $x = -1$
When $x = 2, y = 4$ and $x = -1, y = 1$
Thus the points of intersection of the given curve and line are $(-1, 1)$ and $(2, 4)$.
$\therefore$ Required area=
$$\int_{-1}^{2} (x + 2)dx - \int_{-1}^{2} x^2 dx$$
$$= \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right) - \left(\frac{8}{3} + \frac{1}{3}\right)$$
$$= 6 + \frac{3}{2} - 3 = \frac{9}{2} \text{ sq. units}$$

Q28. If $x = \sin t$, $y = \sin pt$, prove that
$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

S28.
Sol. We have, $x = \sin t$ and $y = \sin pt$
$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = p\cos pt \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$
Differentiating both sides w.r.t. x, we get
$$\frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^2 t} \times \frac{dt}{dx}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^3 t}$$
$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{p^2\sin pt\cos t}{\cos^3 t} + \frac{p\cos pt\sin t}{\cos^3 t}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2y}{\cos^2 t} + \frac{x\frac{dy}{dx}}{\cos^2 t} \Rightarrow \cos^2 t\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$
$$\Rightarrow (1 - \sin^2 t)\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$
$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

Q29. Solve the differential equation:
$(\tan^{-1} y - x)dy = (1 + y^2)dx$.

S29.
Sol. We have, $(\tan^{-1} y - x)dy = (1 + y^2)dx$
$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2} \Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2} \cdot x = \frac{\tan^{-1} y}{1 + y^2}$$
This is a linear differential equation of the form

$\frac{dx}{dy} + Px = 0$, where $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1} y}{1+y^2}$

I.F. $= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

$\therefore$ Required solution is,

$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y} \cdot \tan^{-1} y}{1+y^2} dy + C$ ...(i)

Put $\tan^{-1} y = t \Rightarrow \left(\frac{1}{1+y^2}\right) dy = dt$

$\therefore$ (i) becomes, $x \cdot e^{\tan^{-1} y} = \int e^t \cdot t dt + C$

$\Rightarrow x \cdot e^{\tan^{-1} y} = t \cdot e^t - \int 1 \cdot e^t dt + C$

$\Rightarrow x \cdot e^{\tan^{-1} y} = t \cdot e^t - e^t + C$

$\Rightarrow x \cdot e^{\tan^{-1} y} = \tan^{-1} y \cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$

$\Rightarrow x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$

Q30. A departmental store sends bills to charge its customers once a month. Past experience shows that 70% of its customers pay their first month bill in time. The store also found that the customer who pays the bill in time has the probability of 0.8 of paying in time next month and the customer who doesn't pay in time has the probability of 0.4 of paying in time the next month.
Based on the above information, answer the following question:

(i) Let E_1 and E_2 respectively denote the event of customer paying or not paying the first month bill in time. Find $P(E_1), P(E_2)$.

(ii) Let A denotes the event of customer paying second month's bill in time, then find $P(A|E_1)$ and $P(A|E_2)$.

(iii) Find the probability of customer paying second month's bill in time.

OR

(iii) Find the probability of customer paying first month's bill in time if it is found that customer has paid the second month's bill in time.

S30.

Sol. (i) $P(E_1) = \frac{70}{100} = \frac{7}{10}$ and $P(E_2) = 1 - \frac{7}{10} = \frac{3}{10}$

(ii) $P(A|E_1) = 0.8 = \frac{8}{10}$ and $P(A|E_2) = 0.4 = \frac{4}{10}$.

(iii) $P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)$

[By theorem of total probability]

$= \frac{7}{10} \times \frac{8}{10} + \frac{3}{10} \times \frac{4}{10} = \frac{68}{100} = \frac{17}{25}$

OR

(iii) $P(E_1|A) = \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2)}$

$= \frac{\frac{7}{10} \times \frac{8}{10}}{\frac{7}{10} \times \frac{8}{10} + \frac{3}{10} \times \frac{4}{10}} = \frac{56}{68} = \frac{14}{17}$.